A PROJECTIVE APPROACH TO NON-DIVIDING SETS

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Abstract. Let $F(N)$ be the largest size of a set $A \subseteq \{1, \dots, N\}$ in which no element divides the sum of any nonempty subset of the remaining elements. We prove $F(N) = N^{1/5+o(1)}$, matching the constructions of Erdős. The proof is the density-increment argument of Pham and Zakharov for non-averaging sets, run one dimension lower: divisibility constraints survive the projective normalization $v \mapsto v/\lambda(v)$, which drops the associated convex geometry by one dimension and moves the exponent from $1/4$ to $1/5$. The proof has been formally verified in Lean 4. The novel idea was found by GPT 5.6 Sol.

1. Introduction

For finite $A \subseteq \mathbb{Z}_{>0}$ write $\Sigma(A) = \{\sum_{a\in S} a : S \subseteq A\}$ for its set of subset sums. A finite set $A \subseteq \mathbb{Z}_{>0}$ is *non-dividing* if

$$a \nmid \sum_{s\in S} s \qquad \text{for every } a \in A \text{ and nonempty } S \subseteq A \setminus \{a\},$$

and $F(N)$ denotes the maximum size of a non-dividing subset of $[N] = \{1, \dots, N\}$. Erdős, Lev, Rauzy, Sándor and Sárkőzy introduced the problem and proved $F(N) < 3N^{1/2} + 1$ [4]; it is Problem 131 of [1]. The previously known polynomial estimates were

$$N^{1/5} \ll F(N) \le N^{1/4+o(1)}.$$

The lower bound is implicit in the constructions of [3, 4]. The upper bound holds because every non-dividing set is non-averaging (an averaging relation $\sum_{s\in S} s = |S|a$ is already forbidden modulo a), combined with the sharp non-averaging theorem of Pham and Zakharov [5].

Theorem 1.1. $F(N) = N^{1/5+o(1)}$.

Since the exponent $1/4$ is sharp for non-averaging sets, an improvement must use the divisibilities themselves, and the obstacle is that divisibility is destroyed by translation. Our proof is the density-increment argument of [5] run one dimension lower. A dyadic pigeonhole places the set in a shell $X \le \lambda(v) \le 2X$ of an integer linear functional λ; the forbidden relations $\lambda(a) \mid \sum_{v\in S} \lambda(v)$ see only the values of λ, so they survive passage to subsets and coefficient identifications of proper homogeneous generalized arithmetic progressions (GAPs), the only operations we ever perform; crucially, they also survive the normalization $\pi(v) = v/\lambda(v)$ into the hyperplane $\{\lambda = 1\}$ of dimension $d - 1$. A tube-of-subset-sums argument (Theorem 3.8) forces the projected set into approximate convex position, and the convex-density increment of [5], run in the hyperplane and lifted back to the lattice by Lemma 4.2, moves the exponents from their α_d to $\sigma_d = \alpha_{d-1}$ ($d \ge 2$) and $\sigma_1 = \frac{1}{5}$.

We use [5, Corollary 5, Lemmas 1, 6, 7, 8 and 13] as stated, and their preprocessing, slab and filling steps and exponent numerology in unshifted or projective variants, indicating only the modifications; routine uniformity and bookkeeping verifications are abbreviated. Every adaptation is carried out in full in the Lean 4 formalization [8], whose only non-foundational axioms are the quoted results of [5] and the Blaschke selection and volume-continuity theorems [6].

2. Shell sets, structure theorem, preprocessing

Definition 2.1. Let $B \subseteq \mathbb{Z}^d$ be a symmetric axis-parallel box, $\lambda : \mathbb{Z}^d \to \mathbb{Z}$ a nonzero integer linear functional, and $X > 0$. A finite set $V \subseteq B$ is (λ, X)-*shell non-dividing* if

$$X \le \lambda(v) \le 2X \quad (v \in V), \qquad \text{and} \qquad \lambda(a) \nmid \sum_{v \in S} \lambda(v)$$

for every $a \in V$ and nonempty $S \subseteq V \setminus \{a\}$. For such v put $\pi(v) = v/\lambda(v)$, a point of the affine hyperplane $H_\lambda = \{y \in \mathbb{R}^d : \lambda(y) = 1\}$ of dimension $d-1$. Define

$$\sigma_1 = \tfrac{1}{5}, \qquad \sigma_2 = \tfrac{1}{4}, \qquad \sigma_d = \tfrac{d-2}{d} \quad (d \ge 3);$$

for $d \ge 2$, σ_d is the sharp non-averaging exponent α_{d-1} of [5] in dimension $d-1$.

A symmetric homogeneous GAP is $P = \{\sum_{i=1}^e n_i q_i : |n_i| \le N_i\}$ with $q_i \in \mathbb{Z}^\ell$; its *dimension* is e, its *rank* is $\dim \operatorname{span}_{\mathbb{R}}\{q_1, \dots, q_e\}$, and it is *proper* if the coefficient map is injective. For $t \ge 0$, tP rescales the radii, and for proper P the coefficient identification is $\phi_P : P \to \mathbb{Z}^e$. We write $s(x) = x/(\log x)^2$ and $s(C) = s(|C|)$.

Theorem 2.2 (Conlon–Fox–Pham [2]; [5, Corollary 5])**.** *For every ℓ and $\beta > 1$ there are $c > 0$ and $D \in \mathbb{N}$ such that the following holds for every $C \subseteq \mathbb{Z}^\ell$ of sufficiently large size m lying in an axis-parallel box Q with $|Q| \le m^\beta$. There are a set $\widehat{C} \subseteq C$ with $|\widehat{C}| \ge m - c^{-1}m/\log m$, a symmetric homogeneous GAP $P(C) \supseteq \widehat{C} \cup \{0\}$ of dimension $d' \le D$, and a set $C_0 \subseteq \widehat{C}$ with $|C_0| \le s(C)$, such that*

$$q + cs(C)P(C) \subseteq \Sigma(C_0)$$

for some q, and $cs(C)P(C)$ is proper.

Here automatically $q \in s(C)P(C)$: indeed $q \in \Sigma(C_0) \subseteq s(C)P(C)$, since $C_0 \subseteq P(C)$ and $0 \in cs(C)P(C)$. A *structural package* is a fixed choice, for each eligible C, of the constants and objects in Theorem 2.2, with $P(C)$ of minimum dimension $d(C) = \dim P(C)$; then all radii satisfy $N_i \ge 1$, and for large $|C|$ properness of $cs(C)P(C)$ implies that of $P(C)$. When dimensions range over a fixed finite set, one package per ambient dimension is fixed in advance, so all constants are uniform.

Definition 2.3. Let $V \subseteq B \subseteq \mathbb{Z}^d$ with B a symmetric coordinate box, and fix a structural package. The pair (V, B) is (δ, γ)-*irreducible* if every $D \subseteq V$ with $|D| \ge \delta|V|$ satisfies

$$d(D) = d \qquad \text{and} \qquad |P(D)| \ge \gamma|B|.$$

This is [5, Definition 9] with the shifts of the witnessing subsets omitted.

Lemma 2.4 (Transport of the shell functional)**.** *Let $P \subseteq \mathbb{Z}^\ell$ be a proper homogeneous GAP with steps $q_1, \dots, q_d$ and let λ be integer-linear. Then*

$$\lambda_P(n_1, \dots, n_d) = \sum_{i=1}^d n_i \lambda(q_i)$$

satisfies $\lambda_P(\phi_P(v)) = \lambda(v)$ on P. Consequently, if $V \subseteq P$ is shell non-dividing for λ, then $\phi_P(V)$ is shell non-dividing for λ_P, with the same shell parameter.

Proof. Homogeneity gives $v = \sum n_i q_i$ whenever $\phi_P(v) = (n_1, \dots, n_d)$, and the identity follows by linearity. Every forbidden relation involves only the values of the functional, so it is preserved exactly. $\square$

Lemma 2.5 (Unshifted preprocessing)**.** *Fix ℓ, $\beta > 1$, $\varepsilon \in (0, 1/3)$ and $K_0 \geq 1$. Let $V \subseteq B \subseteq \mathbb{Z}^\ell$ be shell non-dividing with $|B| \leq |V|^\beta$, and let $0 < \delta \leq \frac{1}{2}$, $\gamma \leq \delta^K$ and $\gamma \geq |V|^{-1/3}$, where $K = K(\ell, \beta, \varepsilon, K_0)$ is sufficiently large. Using only subsets and coefficient identifications of proper homogeneous GAPs, one can pass to a shell non-dividing $\widetilde{V} \subseteq \widetilde{B} \subseteq \mathbb{Z}^{\widetilde{d}}$, with a package fixed in $\mathbb{Z}^{\widetilde{d}}$ relative to which $(\widetilde{V}, \widetilde{B})$ is (δ, γ)-irreducible, such that $\widetilde{d} = O_{\ell,\beta}(1)$, the output package has polynomial-box exponent bounded in $\ell, \beta, \varepsilon, K_0$, and*

$$\begin{aligned}
&|\widetilde{B}| \ll |V|^{-(1-\varepsilon)(\widetilde{d}-\ell)}|B|, && |\widetilde{V}| > |V|^{1-\varepsilon}, && \widetilde{d} > \ell,\\
&|\widetilde{B}| \ll \rho^{K_0}|B|, && |\widetilde{V}| = \rho|V| > |V|^{1-\varepsilon}, && \widetilde{d} = \ell,\\
&|\widetilde{B}| \ll |B|, && |\widetilde{V}| > |V|^{1-\varepsilon}, && \widetilde{d} < \ell,
\end{aligned}$$

with implied constants depending on $\ell, \beta, \varepsilon, K_0$.

Proof. This is the unshifted analogue of [5, Lemma 10]. We run their iteration of down-, up- and shrink-moves with three modifications; the accompanying bookkeeping, abbreviated here, is carried out in full in the formalization [8]. Write $n = |V|$ and $\delta_* = \delta/2$. At an accepted stage A_j (with $A_0 = V$) put $P_j = P(A_j)$, $d_j = \dim P_j$, $A_j^\sharp = \phi_{P_j}(\widehat{A}_j)$ and $Q_j = \phi_{P_j}(P_j)$, and stop before accepting a move of size at most $n^{1-\varepsilon/2}$ or of structural dimension exceeding a threshold $\bar{d} > D(\ell, \beta) + 3(\beta + 2) + 1$.

Translations are omitted. A move of [5, Lemma 10] passes to $A_1 - x$ for a witness $A_1 \subseteq A_j^\sharp$ with $|A_1| \geq \delta|A_j^\sharp|$ and a shift $x \in Q_j$; here witnesses are taken with $x = 0$ only, matching Definition 2.3. This restricts the available moves and changes no estimate: the down-, up- and shrink-move bounds

$$|P_{j+1}| \leq C|P_j|, \qquad |P_{j+1}| \leq Cs(A_{j+1})^{-(d_{j+1}-d_j)}|P_j|, \qquad |P_{j+1}| < \gamma|P_j|$$

are supplied by [5, Lemmas 6 and 8], which apply as stated because every box arising is a symmetric coordinate box, and which also give the initial bound $|P_0| \ll n^{-(1-\varepsilon)\max(0,d_0-\ell)}|B| \leq n^{\beta+1}$.

The shell functional is transported. At each coefficient identification the functional is replaced by λ_{P_j} as in Lemma 2.4, so every accepted set remains shell non-dividing with the same shell parameter.

Packages are fixed in advance. Packages with common polynomial-box exponent $\bar{\beta} = 4(\beta + 3)$ are fixed beforehand in all ambient dimensions $r \leq \bar{d}$, and stay applicable throughout: partial products of the move estimates give $|P_j| \leq n^{\beta+2}$, while $\gamma \geq n^{-1/3}$ and $|A_j| > n^{1-\varepsilon/2}$ give $\delta|A_j^\sharp| \geq \frac{1}{2}n^{2/3-\varepsilon/2}$, whose $\bar{\beta}$-th power exceeds $n^{\beta+2}$ because $\varepsilon < 1/3$.

With these modifications the analysis of [5, proof of Lemma 10] applies verbatim. The total upward dimension change is at most $3(\beta + 2)$, so the dimension cutoff never triggers; a proposed iterate of size at most $n^{1-\varepsilon/2}$ would give $\delta_*^{i+1} \leq n^{-\varepsilon/2}$ and, via $\gamma \leq \delta_*^{K/2}$, force $|P_i| < 1$ once K is large, so the size cutoff never triggers either. Hence the procedure stops at an irreducible pair $(\widetilde{V}, \widetilde{B}) = (A_i^\sharp, Q_i)$ with $|\widetilde{V}| > |V|^{1-\varepsilon}$. The box estimates follow from the product bound

$$|P_i| \ll \gamma^{i-|\mathcal{U}|-|\mathcal{D}|}n^{-(1-\varepsilon)M}|B|, \qquad M = U + \max(0, d_0 - \ell),$$

exactly as in loc. cit.: $M \geq \widetilde{d} - \ell$ gives the first case, and discarding factors at most one gives the third; for $\widetilde{d} = \ell$, the bounds $|\mathcal{U}| + |\mathcal{D}| \leq 2M$ and $n^{-(1-\varepsilon)M} \leq \gamma^{2M}$ (valid as $\varepsilon < 1/3$, $\gamma \geq n^{-1/3}$) give $|P_i| \ll \gamma^i|B| \ll_{K_0} \rho^{K_0}|B|$, using $\rho \gg \delta_*^i$, $\gamma \leq \delta_*^{K/2}$ and $K \geq 2K_0$. ☐

3. The tube argument

Throughout this section, $B_\mathbb{R} = \prod_{i=1}^d [-M_i, M_i]$ with $M_i \in \mathbb{Z}_{\geq 1}$ denotes a symmetric real box, $B = B_\mathbb{R} \cap \mathbb{Z}^d$, and $w_B(u) = \max_{x \in B_\mathbb{R}} |u(x)|$ for a linear functional u. The *irreducible* *setting* means: d and $\beta > 1$ are fixed, with a structural package in ambient dimension d of polynomial-box exponent at most β, applicable to every subset of V of size at least $\delta|V|$; the pair (V, B), $V \subseteq B \subseteq \mathbb{Z}^d$, is (δ, γ)-irreducible relative to it; and $n = |V|$ satisfies $|B| \le n^\beta$. For finite $E \subseteq \mathbb{R}^d$ let $Z(E) = \{\sum_{v\in E} t_v v : 0 \le t_v \le 1\}$ be its zonotope, and let $\langle P\rangle$ denote the lattice generated by the steps of a homogeneous GAP P.

Lemma 3.1 (Index and containment)**.** *Assume the irreducible setting, let $C \subseteq V$ satisfy $|C| \ge \delta|V|$ and $|B| \le |C|^\beta$, and put $P = P(C)$, $s = s(C)$, $L = \langle P\rangle$. For all sufficiently large n:*

(i) $P \subseteq \frac{4}{c} B_{\mathbb{R}} \subseteq C_{d,\beta} B_{\mathbb{R}}$;

(ii) *if $s\gamma \to \infty$, then L has rank d and $[\mathbb{Z}^d : L] \ll_{d,\beta} \gamma^{-1}$.*

Proof. Since $C_0 \subseteq B$ and $|C_0| \le s$,

$$q + csP \subseteq \Sigma(C_0) \subseteq sB_{\mathbb{R}};$$

as $0 \in csP$, also $q \in sB_{\mathbb{R}}$, so $csP \subseteq 2sB_{\mathbb{R}}$, and for $p \in P$ and $k = \lfloor cs\rfloor \ge cs/2$, $kp \in csP$ gives $p \in (2s/k)B_{\mathbb{R}} \subseteq (4/c)B_{\mathbb{R}}$. This is the containment step of [5, Lemma 11] and proves (i).

For (ii), irreducibility gives $\dim P = d$ and $|P| \ge \gamma|B|$. Write $P = \{\sum n_i q_i : |n_i| \le N_i\}$ with $N_i \ge 1$, and let $H = \operatorname{span}_{\mathbb{R}}\{q_i\}$ have dimension r. Properness of csP gives $|csP| \gg_{d,\beta} s^d|P|$, while $csP \subseteq H \cap 2sB_{\mathbb{R}}$ and projection to r coordinates on which H injects give $|csP| \le (2s)^r|B|$. Hence

$$s^{d-r}\gamma \ll_{d,\beta} 1,$$

so $r = d$ by $s\gamma \to \infty$, and the steps form a $\mathbb{Z}$-basis of L. With M the matrix of steps,

$$\operatorname{vol}_d(\operatorname{conv} P) = 2^d|\det M| \prod_{i=1}^{d} N_i \asymp_d [\mathbb{Z}^d : L]\,|P|,$$

while (i) gives $\operatorname{vol}_d(\operatorname{conv} P) \ll_{d,\beta} \operatorname{vol}_d(B_{\mathbb{R}}) \asymp_d |B|$; hence $[\mathbb{Z}^d : L]\,|P| \ll_{d,\beta} |B| \le \gamma^{-1}|P|$. □

Lemma 3.2 (Projective span)**.** *Assume the irreducible setting, with V (λ, X)-shell non-dividing, and suppose $[s(\delta n)]^{-1} = o(\gamma)$. For all sufficiently large n,*

$$\operatorname{span}_{\mathbb{R}} V = \mathbb{R}^d, \qquad \operatorname{aff}\pi(V) = H_\lambda, \qquad \pi(V) \subseteq \Omega := \{y \in H_\lambda : Xy \in B_{\mathbb{R}}\},$$

and Ω has nonempty relative interior in H_λ.

Proof. Since $s(V) \ge s(\delta n)$, the hypothesis gives $s(V)\gamma \to \infty$, so Lemma 3.1(ii) with $C = V$ makes the step lattice of $P(V)$ have rank d. For large n every radius of $cs(V)P(V)$ is at least one, so differences of points of $q + cs(V)P(V) \subseteq \Sigma(V_0) \subseteq \operatorname{span}_{\mathbb{R}} V$ realize every step of $P(V)$; hence $\operatorname{span}_{\mathbb{R}} V = \mathbb{R}^d$. If $\pi(V)$ lay in a proper affine subspace of H_λ, then $v = \lambda(v)\pi(v)$ would place V in a proper linear subspace of $\mathbb{R}^d$. Finally $X\pi(v) = \frac{X}{\lambda(v)}v \in B_{\mathbb{R}}$ by the shell condition and convexity of $B_{\mathbb{R}}$, so $\pi(V) \subseteq \Omega$; and Ω is convex and contains an affinely spanning subset of H_λ. □

Lemma 3.3 (Filling)**.** *Let $C \subseteq B \subseteq \mathbb{Z}^d$, let $P = P(C)$ have dimension d and step lattice L of rank d, and put $\overline{C} = \widehat{C} \setminus C_0$ and $s = s(C)$. For all sufficiently large $|C|$,*

$$L \cap \Big(Z(\overline{C}) + q + \frac{cs}{2}P\Big) \subseteq \Sigma(C).$$

Proof. This inclusion is established from the rounding lemma [5, Lemma 13] in the proof of [5, Theorem 4]; we indicate the homogeneous full-rank form. The steps of P are a $\mathbb{Z}$-basis of L, so the coefficient map extends to a linear isomorphism ψ with $\psi(P) = \prod[-N_i, N_i] \cap \mathbb{Z}^d$. Let

$y = z + t$ with $z \in Z(\overline{C})$ and $t \in q + (cs/2)P$; then $z = y - t \in L$, since $q \in sP \subseteq L$. Applying [5, Lemma 13] to $\psi(\overline{C}) \cup \{0\}$ in the coefficient box yields $\widetilde{z} \in \Sigma(\overline{C})$ with

$$z - \widetilde{z} \in 3\sqrt{2d|C|}\, P \subseteq \frac{cs}{2} P$$

for large $|C|$, because $s = |C|/(\log |C|)^2$. Then $(z - \widetilde{z}) + t \in q + csP \subseteq \Sigma(C_0)$, and disjointness of $\overline{C}$ and C_0 gives $y = \widetilde{z} + ((z - \widetilde{z}) + t) \in \Sigma(C)$. □

Lemma 3.4 (Slab exclusion)**.** *Assume the irreducible setting, and suppose $\delta \geq n^{-1/2}$ and $[s(\delta n)]^{-1} = o(\gamma)$. There is $c_{d,\beta} > 0$ such that, for every nonzero linear functional u and every $0 < \tau \leq c_{d,\beta}\gamma$, the central slab $\{x : |u(x)| \leq \tau w_B(u)\}$ contains fewer than δn points of V.*

Proof. This is the unshifted form of the slab claim in the proof of [5, Lemma 14]. Suppose $D \subseteq V$ lies in the slab and $|D| \geq \delta n$. Since $\delta \geq n^{-1/2}$, one has $|B| \leq n^\beta \leq |D|^{2\beta}$, so the package applies to D, and irreducibility gives $d(D) = d$ and $|P(D)| \geq \gamma |B|$.

Let $K = B_{\mathbb{R}} \cap \{x : |u(x)| \leq \tau w_B(u)\}$ and let r be the dimension of $\operatorname{span}_{\mathbb{R}}(K \cap \mathbb{Z}^d)$. The discrete John lemma [5, Lemma 7], applied to K in that span's lattice, places $K \cap \mathbb{Z}^d \supseteq D$ in a proper symmetric rank-r GAP Q with $|Q| \ll_d |K \cap \mathbb{Z}^d|$. If $r < d$, then trivially $|Q| \ll_d |B|$. If $r = d$, then moreover

$$|K \cap \mathbb{Z}^d| \ll_d \operatorname{vol}_d(K) \leq d\tau \operatorname{vol}_d(B_{\mathbb{R}}) \asymp_d \tau |B|,$$

by the successive-minima count [7, Ch. 3] and a section estimate: writing $u(x) = \sum a_i x_i$ and choosing j with $|a_j| M_j \geq w_B(u)/d$, the x_j-sections of K are intervals of length at most $2\tau w_B(u)/|a_j| \leq 2d\tau M_j$.

Put $s_D = s(D) \geq s(\delta n)$. Comparing the proper GAP $q + cs_D P(D) \subseteq \Sigma(D_0) \subseteq s_D Q$ gives $s_D^d |P(D)| \ll_{d,\beta} s_D^r |Q|$. For $r < d$ this yields $|P(D)| \ll s_D^{-1} |B| = o(\gamma |B|)$ by the parameter hypothesis; for $r = d$ it yields $|P(D)| \ll \tau |B| \leq C c_{d,\beta} \gamma |B|$. Choosing $c_{d,\beta}$ small enough contradicts $|P(D)| \geq \gamma |B|$ in either case. □

Lemma 3.5 (Robust weighted zonotope)**.** *Assume the irreducible setting with $\delta \geq n^{-1/2}$, $\delta = o(\mu)$ and $[s(\delta n)]^{-1} = o(\gamma)$. Let $a \in V$, let weights $0 \leq \theta_v \leq \frac{1}{2}$ $(v \neq a)$ satisfy $c_1 \mu n \leq \sum_{v \neq a} \theta_v \leq c_2 \mu n$, let $R \subseteq V \setminus \{a\}$ have size $o(\mu n)$, and put $C = (V \setminus \{a\}) \setminus R$ and $z_C = \sum_{v \in C} \theta_v v$. Then, for a constant $c_{d,\beta,c_1,c_2} > 0$,*

$$z_C + c_{d,\beta,c_1,c_2}\, \mu \gamma n\, B_{\mathbb{R}} \subseteq Z(C).$$

Proof. We follow the proof of [5, Lemma 14], with the slab claim replaced by Lemma 3.4. With $Z^0 = \sum_{v \in C} [-\theta_v, \theta_v] v$ it suffices to prove $\xi n B_{\mathbb{R}} \subseteq Z^0$ for $\xi = c_{d,\beta,c_1,c_2} \mu \gamma$. If $y \in \xi n B_{\mathbb{R}} \setminus Z^0$, separation gives a functional u with

$$z := u(y) > \sum_{v \in C} \theta_v |u(v)|,$$

and $H = \{x : |u(x)| < c_d' \gamma z / (\xi n)\}$ is a central slab of relative width at most $c_d' \gamma$, because $y \in \xi n B_{\mathbb{R}}$. By Lemma 3.4, $|V \cap H| < \delta n$. The mass count is then that of loc. cit.: the total weight is $\asymp \mu n$, removing R costs $o(\mu n)$, and the slab carries at most $\delta n/2 = o(\mu n)$, so $\sum_{v \in C \setminus H} \theta_v \gg \mu n$; since $|u(v)| \geq c_d' \gamma z/(\xi n)$ off H, this gives $\sum_v \theta_v |u(v)| \gg \mu n \cdot \gamma z/(\xi n) > z$ once c_{d,β,c_1,c_2} is small, a contradiction. □

Definition 3.6. An indexed family $p_1, \ldots, p_n$ in a Euclidean affine space is in *μ-convex position* if every p_j lies in a closed half-space containing at most μn indices, counted with multiplicity; this is the indexed variant of the δ-convex position of [5]. In a zero-dimensional space the only nonempty half-space is the whole space.

Lemma 3.7 (Projective duality)**.** *Let V lie in a (λ, X)-shell and suppose the indexed family $\pi(V)$ is not in μ-convex position, where $\mu n \geq 4$. Then there are $a \in V$ and coefficients β_v with*

$$a = \sum_{v \neq a} \beta_v v, \qquad 0 \leq \beta_v \leq \frac{4}{\mu n}, \qquad \frac{1}{3} \leq \sum_{v \neq a} \beta_v \leq 3.$$

Proof. Choose an index a at which convex position fails. The separation argument opening the proof of [5, Theorem 4] yields coefficients $c_v \in [0, (\mu n)^{-1}]$ with $\sum_v c_v = 1$ and $\pi(a) = \sum_v c_v \pi(v)$: a functional separating 0 from the compact convex set of such combinations would be negative on it, while failure of convex position places more than μn indices in the half-space where it is nonnegative, and distributing the unit mass over those indices gives a contradiction. Multiplying by $\lambda(a)$ gives $a = \sum_v c_v \frac{\lambda(a)}{\lambda(v)} v$ with every ratio in $[\frac{1}{2}, 2]$; moving the $v = a$ term to the left and dividing by $1 - c_a \geq \frac{3}{4}$ gives the stated bounds. ☐

Theorem 3.8 (Projective convexity)**.** *Assume the irreducible setting with V (λ, X)-shell non-dividing, $0 < \delta, \gamma, \mu < 1$, and the parameter hierarchy*

$$\delta \geq n^{-1/2}, \qquad \delta = o(\mu), \qquad \frac{n}{\log n} = o(\mu\gamma n), \qquad \gamma^{-1} = o(\mu\gamma n), \qquad [s(\delta n)]^{-1} = o(\gamma).$$

Then $\pi(V)$ is in μ-convex position for all sufficiently large n.

Proof. Suppose not. Lemma 3.7 gives a and coefficients β_v; set $T = \lfloor c_0 \mu n \rfloor$ and $\theta_v = T\beta_v$ for a small absolute constant c_0, so that

$$Ta = \sum_{v \neq a} \theta_v v, \qquad 0 \leq \theta_v \leq \frac{1}{2}, \qquad \sum_{v \neq a} \theta_v \asymp \mu n.$$

Put $C = V \setminus \{a\}$, with structural objects $\widehat{C}, C_0, P, q$, and $L = \langle P \rangle$, $\overline{C} = \widehat{C} \setminus C_0$, $s = s(C)$. For large n one has $|C| \geq \delta n$ and $|B| \leq |C|^{2\beta}$, so Lemma 3.1 (applied with exponent 2β) gives $\operatorname{rank} L = d$, $[\mathbb{Z}^d : L] \ll_{d,\beta} \gamma^{-1}$ and $P \subseteq C_{d,\beta} B_{\mathbb{R}}$.

The discarded set $R = C \setminus \overline{C}$ has size $O(n/\log n) = o(\mu n)$, so Lemma 3.5 gives

$$\overline{z} + \xi_0 n B_{\mathbb{R}} \subseteq Z(\overline{C}), \qquad \overline{z} = \sum_{v \in \overline{C}} \theta_v v, \qquad \xi_0 \asymp_{d,\beta} \mu\gamma,$$

while $Ta - \overline{z} \in O(n/\log n) B_{\mathbb{R}}$ and $q \in sP$. Hence $(Ta - \overline{z}) - q \in O(n/\log n + s) B_{\mathbb{R}} = o(\xi_0 n) B_{\mathbb{R}}$ by the hierarchy, so with $\xi = \xi_0/2$, for large n, $(Ta - \overline{z}) - q + \xi n B_{\mathbb{R}} \subseteq \xi_0 n B_{\mathbb{R}}$. Consequently every $y \in L \cap (Ta + \xi n B_{\mathbb{R}})$ satisfies $y - q \in \overline{z} + \xi_0 n B_{\mathbb{R}} \subseteq Z(\overline{C})$, hence $y \in Z(\overline{C}) + q + \frac{cs}{2} P$, and Lemma 3.3 yields the tube inclusion

$$L \cap (Ta + \xi n B_{\mathbb{R}}) \subseteq \Sigma(C). \tag{†}$$

Let r be the order of $a + L$ in $\mathbb{Z}^d / L$; then $r \ll_{d,\beta} \gamma^{-1}$, so $r = o(\xi n)$ and $r = o(T)$ by the hierarchy. The nearest multiple m_0 of r to T is therefore positive and satisfies $|m_0 - T| \leq r/2 < \xi n$; since $a \in B$,

$$m_0 a \in L \cap (Ta + \xi n B_{\mathbb{R}}) \subseteq \Sigma(V \setminus \{a\})$$

by (†). Applying λ gives $m_0 \lambda(a) = \sum_{v \in S} \lambda(v)$ for a nonempty $S \subseteq V \setminus \{a\}$, contradicting shell non-divisibility. ☐

4. Projective density increment

Lemma 4.1 (Weighted convex-density lemma)**.** *Fix $r \geq 1$ and $\varepsilon > 0$. There are $\tau \in (0, 1)$ and $\mu_0 > 0$ such that the following holds for $0 < \mu < \mu_0$ and n sufficiently large in r, ε, μ. If an* *indexed family $p_1, \dots, p_n$ in a convex body $\Omega \subseteq \mathbb{R}^r$ with nonempty interior is in μ-convex position, then there are $\eta \in [\mu, \mu^\tau]$ and a compact convex $\Omega' \subseteq \Omega$ with*

$$\mathrm{vol}_r(\Omega') \leq \eta \, \mathrm{vol}_r(\Omega), \qquad \#\{i : p_i \in \Omega'\} \geq \eta^{\kappa_r + \varepsilon} n, \qquad \kappa_r = \frac{r-1}{r+1}.$$

Proof. Take τ and μ_0 from [5, Lemma 1]. For each j choose a witnessing closed half-space H_j containing p_j and at most μn indices, and for $m \geq 1$ perturb the points successively to distinct $q_i^{(m)} \in H_i \cap B(p_i, 1/m)$, so close to p_i that $q_i^{(m)} \in H_j$ only when $p_i \in H_j$; this is possible because only finitely many half-spaces H_j omit p_i, each at positive distance, and $H_i \cap B(p_i, 1/m)$ is infinite as $r \geq 1$. The perturbed set is in μ-convex position inside $\Omega_m = \Omega + m^{-1} \overline{B_2^r}$, so [5, Lemma 1] gives $\eta_m \in [\mu, \mu^\tau]$ and a compact convex $K_m \subseteq \Omega_m$ with $\mathrm{vol}_r(K_m) \leq \eta_m \, \mathrm{vol}_r(\Omega_m)$ and $|\{i : q_i^{(m)} \in K_m\}| \geq \eta_m^{\kappa_r + \varepsilon} n$.

Pass to a subsequence along which $\eta_m \to \eta$, the finite index set $\{i : q_i^{(m)} \in K_m\}$ is constant, and $K_m \to K$ in the Hausdorff metric (Blaschke selection; convex-body volume is continuous under Hausdorff convergence [6]). Since Ω_m decreases to Ω, one has $K \subseteq \Omega$, every stabilized index satisfies $p_i \in K$, and $\Omega' = K$ has the required properties. □

Lemma 4.2 (Cap lifting)**.** *Let $B_{\mathbb{R}} \subseteq \mathbb{R}^d$ be a symmetric coordinate box, $B = B_{\mathbb{R}} \cap \mathbb{Z}^d$, λ a nonzero integer linear functional, m_λ the induced Euclidean measure on H_λ, and $\Omega = \{y \in H_\lambda : Xy \in B_{\mathbb{R}}\}$. Let $\Omega' \subseteq \Omega$ be compact convex with $m_\lambda(\Omega') \leq \eta \, m_\lambda(\Omega)$, and put*

$$W = \{v \in B : X \leq \lambda(v) \leq 2X, \ \pi(v) \in \Omega'\}.$$

Then there are a symmetric coordinate box $B' \subseteq \mathbb{Z}^r$, an integer-linear functional λ', and an injective map $\phi : W \to B'$ with $\lambda'(\phi(v)) = \lambda(v)$ on W, such that either $r < d$ and $|B'| \ll_d |B|$, or $r = d$ and $|B'| \ll_d \eta |B|$. In particular, shell non-divisibility and the shell parameter are preserved exactly.

Proof. Write $\lambda(x) = \ell \cdot x$ and, for $E \subseteq H_\lambda$ and $T > 0$, put $\mathrm{Cone}_T(E) = \{ty : y \in E, \ 0 \leq t \leq T\}$. Integrating the Jacobian gives

$$\mathrm{vol}_d(\mathrm{Cone}_T(E)) = \frac{T^d}{d\|\ell\|_2} \, m_\lambda(E),$$

so the λ-dependent factor cancels from volume ratios. Let $K = B_{\mathbb{R}} \cap \mathrm{Cone}_{2X}(\Omega')$, a convex set containing 0 and W; since $\mathrm{Cone}_X(\Omega) \subseteq B_{\mathbb{R}}$, the display gives $\mathrm{vol}_d(K) \leq 2^d \eta \, \mathrm{vol}_d(B_{\mathbb{R}})$.

Put $S = (K - K) \cap \mathbb{Z}^d$, $H = \mathrm{span}_{\mathbb{R}} S$ and $r = \dim H$; then $W \subseteq S$ and $K - K \subseteq 2B_{\mathbb{R}}$. The discrete John lemma [5, Lemma 7], applied to $(K - K) \cap H$ in the lattice $H \cap \mathbb{Z}^d$, gives a proper symmetric homogeneous rank-r GAP $Q \supseteq S$ with $|Q| \ll_d |S|$. If $r < d$, then $|S| \leq |2B_{\mathbb{R}} \cap \mathbb{Z}^d| \leq 2^d |B|$. If $r = d$, the successive-minima count [7, Ch. 3] and the Rogers–Shephard inequality [6] give

$$|S| \ll_d \mathrm{vol}_d(K - K) \leq \binom{2d}{d} \mathrm{vol}_d(K) \ll_d \eta |B|.$$

Finally let B' be the coefficient box of Q, let $\phi = \phi_Q|_W$, and let $\lambda'(n_1, \dots, n_r) = \sum n_i \lambda(q_i)$; Lemma 2.4 gives $\lambda'(\phi(v)) = \lambda(v)$, and $|B'| = |Q|$ satisfies the stated bounds. □

5. The iteration and the proof of Theorem 1.1

Lemma 5.1 (Numerology)**.** *Fix $D \geq 2$ and $\zeta_0 > 0$. There are $\varepsilon_0, c_* > 0$ such that, whenever $1 \leq d < e \leq D$, $\zeta_0 \leq \zeta \leq 1 - \sigma_d$ and $0 < \varepsilon < \varepsilon_0$,*

$$(\sigma_e + \zeta)\Big(\frac{1}{\sigma_d + \zeta} - (1 - \varepsilon)(e - d)\Big) \leq 1 - c_*$$

whenever the parenthetical factor is nonnegative.

Proof. For $2 \le d < e$ this is [5, Observation 15] with all dimensions shifted down by one, the gap depending continuously on ζ. Only the pairs with $d = 1$ are new: for $k = e - 1$, the function

$$f_{1,k}(\zeta) = (\sigma_{1+k} + \zeta)\Big(\frac{1}{\sigma_1 + \zeta} - k\Big)$$

satisfies $f_{1,k}(0) = \sigma_{1+k}(5-k) \le 1$, with equality only for $e \in \{2, 3, 4\}$, and $f'_{1,k}(\zeta) = \frac{\sigma_1 - \sigma_{1+k}}{(\sigma_1+\zeta)^2} - k < 0$, so the gap is strictly positive for $\zeta \ge \zeta_0$. Compactness over the finitely many pairs and the range $\zeta \in [\zeta_0, 1 - \sigma_d]$ gives a uniform $c_* > 0$, which persists for ε_0 small. $\square$

Theorem 5.2. *For fixed d and $\beta > 1$, every (λ, X)-shell non-dividing set $V \subseteq B \subseteq \mathbb{Z}^d$, with B a symmetric coordinate box and $|B| \le |V|^\beta$, satisfies $|V| \le |B|^{\sigma_d + o(1)}$, uniformly in λ and X.*

Proof. Suppose for contradiction that for some $\zeta_0 > 0$ there are arbitrarily large examples $A_0 \subseteq B_0 \subseteq \mathbb{Z}^{d_0}$ with $|A_0| > |B_0|^{\sigma_{d_0} + \zeta_0}$; since $|A_0| \le |B_0|$, we may assume $\sigma_{d_0} + \zeta_0 < 1$. All sets constructed below remain shell non-dividing, by Lemma 2.4.

Choice of parameters. At any stage of excess $\zeta \ge \zeta_0$ one has $|B| < |A|^{\beta_*}$ with $\beta_* = (\sigma_1 + \zeta_0)^{-1}$, and the stage dimension is at most $D_0 = \max\{r : \sigma_r \le 1 - \zeta_0\}$. Choose D above D_0 and above every output-dimension bound of Lemma 2.5 for dimensions $\le D_0$ with exponent β_*; let ε_0, c_* be as in Lemma 5.1 for (D, ζ_0), and put $\Delta_D = \min_{2 \le r \le D}(\sigma_r - \sigma_{r-1})$. Choose

$$\varepsilon < \min\Big\{\varepsilon_0, \frac{\zeta_0}{4}, \frac{c_*}{4}, \frac{\Delta_D}{8}, \frac{1}{3}\Big\} \quad \text{with} \quad \kappa_{r-1} + \varepsilon < \sigma_r + \zeta_0 \quad (2 \le r \le D),$$

choose K_0 with $K_0^{-1} < \sigma_1 + \zeta_0$ and $K_0(\sigma_1 + \zeta_0) > 2$, let K be admissible in Lemma 2.5 for all $\ell \le D_0$, let $\beta^\dagger$ bound the output package exponents and $2\beta_*/(1-\varepsilon)$, and fix a small $\kappa \in (0, 1)$. With $N_r(t)$ a largeness threshold for Lemma 4.1 in dimension r at parameter $\mu = t$, choose a nondecreasing $\omega(n) \to \infty$ with

$$\omega(n)^{2K+2} = o(\log n), \qquad N_r(\omega(n)^{-\kappa}) \le n^{1-\varepsilon} \quad (1 \le r \le D-1).$$

At a stage of size n use $\delta = \omega(n)^{-1}$, $\gamma = \delta^K$, $\mu = \delta^\kappa$; then, uniformly,

$$\delta \ge n^{-1/2}, \quad \gamma \ge n^{-1/3}, \quad \delta = o(\mu), \quad \frac{1}{\log n} = o(\mu\gamma), \quad \gamma^{-1} = o(\mu\gamma n), \quad [s(\delta n)]^{-1} = o(\gamma),$$

routine verifications carried out in the formalization [8].

The one-step increment. At a stage $A \subseteq B \subseteq \mathbb{Z}^d$ with $n = |A|$, let $p = \frac{\log n}{\log |B|} = \sigma_d + \zeta$, so $\zeta_0 \le \zeta \le 1 - \sigma_d$ and $d \le D_0$. Apply Lemma 2.5: the output $\widetilde{A} \subseteq \widetilde{B} \subseteq \mathbb{Z}^e$ has $\widetilde{n} = |\widetilde{A}| > n^{1-\varepsilon}$, the displayed parameter conditions persist with $\widetilde{n}$ in place of n, $|\widetilde{B}| \le \widetilde{n}^{\beta^\dagger}$, and $\widetilde{n} \ge N_r(\mu)$ for all $r < D$.

The dimension-changing cases are handled exactly as Cases 1–2 of the proof of [5, Theorem 2], with [5, Observation 15] replaced by Lemma 5.1. If $e > d$, put $a = \frac{1}{p} - (1-\varepsilon)(e-d)$; the box estimate gives $|\widetilde{B}| \ll n^a$, impossible for large n when $a \le 0$, and otherwise $(\sigma_e + \zeta)a \le 1 - c_*$ yields

$$|\widetilde{A}| > n^{1-\varepsilon} > |\widetilde{B}|^{\sigma_e + \zeta + c_{\mathrm{up}}}$$

for a fixed $c_{\mathrm{up}} > 0$; any noncontradictory output has $\sigma_e + \zeta + c_{\mathrm{up}} < 1$, hence $e \le D_0$. If $e < d$, then $|\widetilde{B}| \ll |B|$ and $\sigma_d - \sigma_e \ge \Delta_D$ give

$$|\widetilde{A}| > |\widetilde{B}|^{\sigma_e + \zeta + c_{\mathrm{down}}}$$

for a fixed $c_{\mathrm{down}} > 0$.

Suppose $e = d$, and write $|\widetilde{A}| = \rho n$ and $|\widetilde{B}| \ll \rho^{K_0}|B|$. If $d = 1$, Theorem 3.8 is already contradictory: the hyperplane $H_{\widetilde{\lambda}}$ is zero-dimensional, its only nonempty half-space contains every index, and $\mu < 1$. Let $d \geq 2$. By Lemma 3.2, the family $\pi(\widetilde{A})$ affinely spans $H_{\widetilde{\lambda}}$ and lies in the convex body Ω, which has nonempty relative interior. Theorem 3.8, Lemma 4.1 applied in $H_{\widetilde{\lambda}} \cong \mathbb{R}^{d-1}$, and Lemma 4.2 then produce a shell non-dividing $A' \subseteq B'$ with

$$|A'| \geq \rho\eta^{a_d}n, \qquad a_d = \kappa_{d-1} + \varepsilon, \qquad \eta \in [\mu, \mu^{\widehat{\tau}_d}], \qquad \widehat{\tau}_d = \tau_{d-1} > 0.$$

If cap lifting lowers the dimension to $f < d$, then $|B'| \ll |\widetilde{B}| \ll |B|$ and $|A'| \geq n^{1-2\varepsilon}$ (as $\eta^{a_d} = n^{-o(1)}$), and the computation of the case $e < d$ gives $|A'| > |B'|^{\sigma_f+\zeta+c_{\mathrm{down}}}$. If it preserves the dimension, then $|B'| \ll \eta\rho^{K_0}|B|$ and

$$\frac{|A'|}{|B'|^p} \gg \eta^{a_d-p}\rho^{1-K_0p}.$$

Uniformly for $2 \leq d \leq D$ one has $p - a_d \geq \nu_0 > 0$ and $K_0p \geq 2$, so with $\tau_* = \min_{2\leq r\leq D}\widehat{\tau}_r$ the right-hand side is at least $C_D^{-1}\mu^{-\tau_*\nu_0} \to \infty$: the excess strictly increases, and moreover $|B'|/|B| \ll_D \eta\rho^{K_0} \leq C_D\mu^{\tau_*} = o(1)$.

Termination. Upward steps increase the excess by c_{up} while the excess stays below 1, so there are $O_{D,\zeta_0}(1)$ of them; downward steps decrease the dimension, so there are at most $M = M(D, \zeta_0)$ dimension-changing steps in all, each retaining at least $|A_j|^{1-2\varepsilon}$ points. For a maximal block of same-dimensional steps in dimension r, put $c_r = \max\{\kappa_{r-1} + \varepsilon, K_0^{-1}\} < \sigma_r + \zeta_0$ and, at step j, $x_j = \log(1/\eta_j)$, $y_j = \log(1/\rho_j)$. The point and box estimates give

$$\log\frac{|A_j|}{|A_{j+1}|} \leq (\kappa_{r-1} + \varepsilon)x_j + y_j \leq c_r(x_j + K_0y_j), \qquad \log\frac{|B_j|}{|B_{j+1}|} \geq x_j + K_0y_j - O_D(1),$$

where the $O_D(1)$ term is $o(x_j+K_0y_j)$, because $x_j \geq \tau_*\kappa\log\omega(|A_j|) \to \infty$ as long as all cardinalities exceed a fixed power of $n_0 = |A_0|$. Summing over any initial segment of the block,

$$\log\frac{|A_{\mathrm{start}}|}{|A_{\mathrm{end}}|} \leq (c_r + o(1))\log|B_{\mathrm{start}}|, \qquad \text{whence} \qquad |A_{\mathrm{end}}| \geq |A_{\mathrm{start}}|^{1-c_r/(\sigma_r+\zeta_0)-o(1)} \geq |A_{\mathrm{start}}|^{\theta}$$

for a fixed $\theta > 0$. With at most M dimension changes and $M + 1$ blocks, every iterate satisfies $|A_j| \geq n_0^q$ for a fixed $q > 0$; running the construction with the provisional stopping rule $|A_j| < n_0^{q/2}$ makes the $o(1)$ terms uniform and shows the rule never triggers. The iteration therefore never stops; but after the last dimension change every step multiplies the box cardinality by a factor $o(1) < \frac{1}{2}$, while the sets keep at least $n_0^q > 1$ elements. This is the required contradiction. □

Proof of Theorem 1.1. The lower bound $F(N) \gg N^{1/5}$ is implicit in the constructions of [3, 4] (see also [1]): translating a Bosznay-type non-averaging set of size m and diameter D by any $x > mD - \min B$ turns divisibility relations into averaging relations, and optimizing the parameters places such a set in $[2m^5]$.

For the upper bound, fix $0 < \varepsilon < 1$ and suppose, for arbitrarily large N, that some non-dividing $A \subseteq [N]$ has $|A| > N^{1/5+2\varepsilon}$. Some dyadic interval $[X, 2X)$ contains $A_0 \subseteq A$ with $|A_0| \geq |A|/(1 + \log_2 N) > N^{1/5+\varepsilon}$. Take $d = 1$, $B = [-2X, 2X] \cap \mathbb{Z}$ and $\lambda(t) = t$; then A_0 is shell non-dividing, $|B| \leq 4N + 1$, and for large N,

$$|A_0| > |B|^{1/5+\varepsilon/2}, \qquad |B| < |A_0|^{\beta}, \qquad \beta = (1/5 + \varepsilon/2)^{-1} > 1.$$

Theorem 5.2 gives $|A_0| \leq |B|^{1/5+o(1)}$, a contradiction for large N. □

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